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SOME CONTACT PROBLEMS IN STEADY-STATE NONLINEAR CREEP IN CASES WITH THIN COVERINGS

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1. We shall first give some fundamental relations in the nonlinear theory of creep for the case of plane deformation which are necessary for the rest of our discussion [1]:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{1.1}$$
$$\frac{\partial^2 \dot{\varepsilon}_x}{\partial y^2}+\frac{\partial^2 \dot{\varepsilon}_y}{\partial x^2}=2\frac{\partial^2 \dot{\gamma}_{xy}}{\partial x\,\partial y},$$
$$\dot{\varepsilon}_x=A\sigma_i^{m-1}[(1-\nu)\sigma_x-\nu\sigma_y],\ \dot{\varepsilon}_y=A\sigma_i^{m-1}[(1-\nu)\sigma_y-\nu\sigma_x],$$
$$\dot{\gamma}_{xy}=A\sigma_i^{m-1}\tau_{xy}\quad(m\geqslant 1),$$
$$\sigma_i=\frac{1}{\sqrt{6}}\sqrt{(\sigma_x-\sigma_y)^2+[(1-\nu)\sigma_x-\nu\sigma_y]^2+[(1-\nu)\sigma_y-\nu\sigma_x]^2+6\tau_{xy}^2},$$

where A is the creep modulus and ν is the Poisson coefficient.

Now we consider the solutions of some problems in the equilibrium of a thin layer† $(|x|<\infty,\ 0\leqslant y\leqslant h)$, whose physical and mechanical properties can be described by the system of equations (1.1). Suppose that the boundary conditions on the faces of the layer have the form

$$\tau_{xy}=0\ (y=0,\ y=h),\ \sigma_y=-p^*(x,\ t)\quad(y=h), \tag{1.2}$$
$$p^*(x,t)=p(x,t)\quad(|x|\leqslant a),\ p^*=0\quad(|x|>a),\ \dot{v}=B\dot{\sigma}_y\quad(y=0).$$

Here t is time; v is displacement along the y axis; B is some linear operator whose form will be indicated below, or

$$\tau_{xy}=0\ (y=h),\ \sigma_y=-p^*(x,\ t)\ (y=h), \tag{1.3}$$
$$\dot{u}=0\,(y=0),\ \dot{v}=B\dot{\sigma}_y\quad(y=0),$$

u is displacement along the x axis.

Taking account, furthermore, of the fact that the layer is thin, we see that instead of the condition of compatibility of the rates of deformation defined by the third formula in (1.1), we can take

$$\tau_{xy}=f_1(x)+yf_2(x). \tag{1.4}$$

Then the approximate solutions of the boundary-value problems (1.1)-(1.4) can be written in the form [2, 3]

$$\tau_{xy}=\sigma_x=0,\ \sigma_y=-p^*(x,\ t), \tag{1.5}$$
$$\dot{\varepsilon}_y=-A(1-\nu)[(1-\nu+\nu^2)/3]^{(m-1)/2}|p^*(x,t)|^m\operatorname{sgn}p^*(x,t);$$
$$\sigma_x=-\nu(1-\nu)^{-1}p^*(x,t),\ \sigma_y=-p^*(x,t),\ \tau_{xy}=-\nu(1-\nu)^{-1}\times$$
$$\times(h-y)[p^*(x,t)]',\ \dot{\varepsilon}_y=-A3^{(1-m)/2}\left[(1-2\nu)(1-\nu)^{-1}\right]^m|p^*(x,t)|^m\operatorname{sgn}p^*(x,t). \tag{1.6}$$

†A layer will be considered thin if the length $2a$ of its actively loaded segment is small in comparison with the thickness h.

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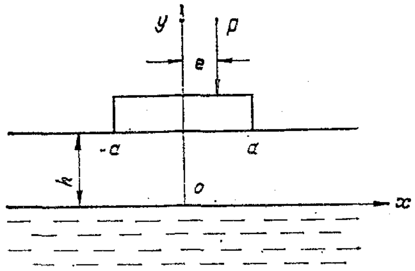

Fig. 1

Using (1.5) and (1.6), we can satisfy the last boundary conditions of (1.2) and (1.3); we will then have, respectively, for y = k and $p^*(x, t) \geqslant 0$:

$$v^{\cdot} = -Ah(1-\nu)[(1-\nu+\nu^2)/3]^{(m-1)/2}[p^*(x, t)]^m - B[p^*(x, t)]^{\cdot}; \quad (1.7)$$

$$v^{\cdot} = -Ah3^{(1-m)/2}[(1-2\nu)(1-\nu)^{-1}]^m [p^*(x, t)]^m - B[p^*(x, t)]^{\cdot}. \quad (1.8)$$

2. Making use of the formulas of Sec. 1, we consider the following contact problem concerning frictionless indentation by a force P(t) (the eccentricity of application of the force is equal to e with respect to the center of the line of contact $|x| \leqslant a$ (Fig. 1)) into the surface of a layer, lying on the hydraulic base ($y \leqslant 0$), of a rigid die. Outside of the die the surface of the layer is free of load, and by virtue of the contact condition for y = h under the die

$$v = -[\delta(t) + \alpha(t)x - f(x)] \quad (|x| \leqslant a), \quad (2.1)$$

where $\delta(t) + \alpha(t)x$ is its rigid displacement; f(x) is the shape of the die base. We shall also assume that during the process of quasistatic indentation of the die into the layer there is no break in the contact between its lower face and the liquid.

Using relation (1.7) and assuming that in this relation $B = (\rho g)^{-1}$, since for the hydraulic base when y = 0, we have $\sigma_y = \rho g v$ (ρ is the density of the liquid, g is the acceleration of gravity), and also including in our calculation the fact that when $|x| \leqslant a$, the function v is given by the relation (2.1) and $p^*(x, t) = p(x, t)$, we obtain an equation for determining the contact pressure p(x, t):

$$\begin{gathered} hAC[p(x, t)]^m + (\rho g)^{-1}[p(x, t)]^{\cdot} = \delta^{\cdot}(t) + \alpha^{\cdot}(t)x \quad (t > 0), \\ (\rho g)^{-1}p(x, 0) = \delta(0) + \alpha(0)x - f(x) \quad (t = 0), \\ C = (1-\nu)[(1-\nu+\nu^2)/3]^{(m-1)/2}. \end{gathered} \quad (2.2)$$

To Eq. (2.2) we should also adjoin the condition for equilibrium of the die:

$$P(t) = \int_{-a}^{a} p(x, t)\,dx, \quad P(t)e = \int_{-a}^{a} xp(x, t)\,dx. \quad (2.3)$$

In dimensionless variables, taking account of the notation

$$\begin{gathered} xa^{-1} = x', \ ea^{-1} = e', \ \lambda = ha^{-1}, \ tt_0^{-1} = t', \ D = t_0\lambda AC(a\rho g)^m, \\ \delta(t)a^{-1} = \gamma(t'), \ \varphi(x', t') = p(x, t)(a\rho g)^{-1}, \ N(t') = P(t)(a^2\rho g)^{-1} \end{gathered}$$

(t_0 is the time scale, and the primes in x' and y' will be omitted from here on), formulas (2.2) and (2.3) take the form

$$\begin{gathered} D[\varphi(x, t)]^m + \varphi^{\cdot}(x, t) = \gamma^{\cdot}(t) + \alpha^{\cdot}(t)x, \ \varphi(x, 0) = \gamma(0) + \alpha(0)x \\ - f(x), \ N(t) = \int_{-1}^{1} \varphi(x, t)\,dx, \ N(t)e = \int_{-1}^{1} x\varphi(x, t)\,dx. \end{gathered} \quad (2.4)$$

From this point on, we shall consider, without loss of generality, the case of a plane die ($\alpha(t) = e \equiv 0$) and assume that in a neighborhood of t = 0 we are given an expansion for the force

$$N(t) = N_0 + N_1 t + \ldots + N_n t^n + O(t^{n+1}). \quad (2.5)$$

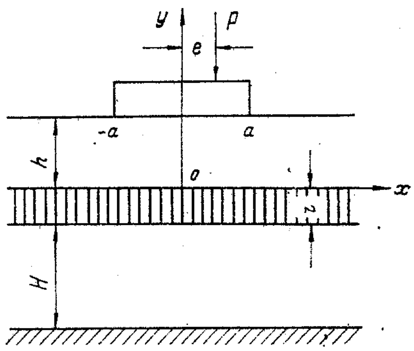

Fig. 2

From system (2.4) we must determine the functions $\varphi(x, t)$ and $\gamma(t)$. We shall seek them in the form

$$\varphi(x,\ t) = \varphi_0[1 + \varphi_1 t + \ldots + \varphi_n t^n + O(t^{n+1})],\ \varphi_0 = \gamma(0), \quad (2.6)$$
$$\gamma(t) = \delta + \gamma_1 t + \gamma_2 t^2 + \ldots + \gamma_n t^n + O(t^{n+1}),\ \delta = \gamma(0).$$

Substituting (2.5) and (2.6) into Eqs. (2.4) and equating in the resulting formulas the terms on the left and right sides with equal powers of t, we obtain

$$\varphi_0 = \frac{1}{2}N_0,\ \varphi_n = \frac{1}{2}\varphi_0^{-1}N_n \quad (n = 1, 2, \ldots), \quad (2.7)$$
$$\gamma_1 = D\varphi_0^m + \varphi_1,\ \gamma_2 = \frac{1}{2}Dm\varphi_0^m\varphi_1 + \varphi_2, \ldots$$

Thus, formulas (2.6) and (2.7) define the asymptotic behavior of the solution of the problem for small values of time.

Now suppose that we are given the expansion for indenting the force in a neighborhood of $t = \infty$:

$$N(t) = N_0 + N_1 e^{-Dt} + \ldots + N_n e^{-Dnt} + O(e^{-D(n+1)t}). \quad (2.8)$$

We shall seek the functions $\varphi(x, t)$ and $\gamma(t)$ satisfying the system of equations (2.4) in the form

$$\varphi(x,\ t) = \varphi_\infty[1 + \varphi_1 e^{-Dt} + \ldots + \varphi_n e^{-Dnt} + O(e^{-D(n+1)t})], \quad (2.9)$$
$$\gamma(t) = \gamma_1 t + \gamma_0 + \beta_1 e^{-Dt} + \ldots + \beta_n e^{-Dnt} + O(e^{-D(n+1)t})].$$

Here we can approximately consider $\gamma_0 = \frac{1}{2}N_0 + \sum_{j=1}^{\infty}\left(\frac{1}{2}N_j - \beta_j\right)$. In addition, a more exact algorithm for determining γ_0 will be described in Sec. 4.

Substituting (2.8) and (2.9) into (2.4) and equating the coefficients of equal powers of e^{-Dt} on the right and left sides, we obtain

$$\varphi_\infty = (1/2)N_0,\ \varphi_n = (1/2)\varphi_\infty^{-1}N_n \quad (n = 1, 2, \ldots), \quad (2.10)$$
$$\gamma_1 = D\varphi_\infty^m,\ \beta_1 = \varphi_1(\varphi_\infty - m\varphi_\infty^m),\ \beta_2 = \varphi_2(\varphi_\infty - (1/2)m\varphi_\infty^m), \ldots$$

Formulas (2.9) and (2.10) define the asymptotic behavior of the solution of the above problem for large values of time. It should be noted that when N = const, it follows that $\varphi = (1/2)N$, $\gamma = (1/2)N + D((1/2)N)^m t$.

3. We consider the case in which a layer whose rheological properties are described by Eqs. (1.1) lies on a rod base $(-l \leqslant y \leqslant 0)$, which in turn is supported by a rigid base. In other respects the formulation of the problem given in Sec. 2 remains unchanged.

The rod base for the case of plane deformation is described by the equation

$$u = 0,\ v = v(y),\ \varepsilon_x = 0,\ \varepsilon_y = R\sigma_y,\ \gamma_{xy} = 0, \quad (3.1)$$
$$R = (1 - 2\mu)[2q(1 - \mu)]^{-1},\ \sigma_x = \mu(1 - \mu)^{-1}\sigma_y,\ \tau_{xy} = 0,\ v'' = 0,$$

where q is the shear modulus and μ is the Poisson coefficient of the rod material.

From formulas (3.1) we find that when y = 0,

$$v = Rl\sigma_y. \tag{3.2}$$

If we now substitute into relation (1.8) when $|x| \leqslant a$ the functions $p^*(x, t) = p(x, t)$ and $v^{\cdot}$ in the form (2.1) and set $B = Rl$ in it, according to (3.2), we obtain the following equation for determining the contact pressure p(x, t):

$$hAC[p(x, t)]^m + Rl\dot{p}(x, t) = \delta^{\cdot}(t) + \alpha^{\cdot}(t)x \quad (t > 0), \tag{3.3}$$

$$Rlp(x, 0) = \delta(0) + \alpha(0)x - f(x) \quad (t = 0),\ C = 3^{(1-m)/2}[(1 - 2\nu)(1 - \nu)^{-1}]^m.$$

To Eqs. (3.3) we must adjoin the statics conditions (2.3).

Asymptotic solutions for the system of equations (3.3), (2.3) for small and large values of time can be obtained according to the scheme described in Sec. 2.

Now we consider the case in which a thin layer lies on a two-layer base. This is (Fig. 2) an elastic layer $(-H - l \leqslant y \leqslant -l)$ lying on a rigid base and covered with a rod layer $(-l \leqslant y \leqslant 0)$. In other respects the formulation of the problems described at the beginning of Sec. 2 is retained.

For an elastic layer in the case when it undergoes plane deformation† with $y = -l$, we have [4]

$$v = \frac{1}{\pi\theta} \int_{-\infty}^{\infty} \sigma_y |_{y=-l} K\left(\frac{\xi - x}{H}\right) d\xi \quad \left(\theta = \frac{4(1-\sigma)G}{3 - 4\sigma}\right), \tag{3.4}$$

$$K(z) = \int_0^{\infty} L(\zeta) \cos \zeta z d\zeta \left(z = \frac{\xi - x}{H}\right),$$

where G is the shear modulus and σ is the Poisson coefficient of the elastic layer.

It is known that: 1) the function L(ζ) is continuous, real, and even on the real axis;

$$2)\ L(\zeta) > 0 \quad (|\zeta| < \infty);$$

$$3)\ L(\zeta)\zeta = A_1\zeta + O(\zeta^3)(\zeta \to 0), L(\zeta)\zeta = 1 + O\left(e^{-A_2\zeta}\right) \quad (\zeta \to \infty). \tag{3.5}$$

Taking account of the fact that σ_y remains constant across the thickness of the rod layer and making use of formula (3.1), we find for y = 0 that

$$v = Rl\sigma_y + \frac{1}{\pi\theta} \int_{-\infty}^{\infty} \sigma_y K\left(\frac{\xi - x}{H}\right) d\xi. \tag{3.6}$$

As before, we substitute into the relation (1.8) when $|x| \leqslant a$ the functions $p^*(x, t) = p(x, t)$ and $v^{\cdot}$ in the form (2.1) and, according to (3.6), setting

$$B(\ldots) = Rl(\ldots) + \frac{1}{\pi\theta} \int_{-\infty}^{\infty} (\ldots) K\left(\frac{\xi - x}{H}\right) d\xi,$$

in that relation, we obtain an integral equation for determining the unknown contact pressure p(x, t) under the die:

$$hAC[p(x, t)]^m + Rlp^{\cdot}(x, t) + \frac{1}{\pi\theta} \int_{-a}^{a} p^{\cdot}(\xi, t) K\left(\frac{\xi - x}{H}\right) d\xi\ \delta^{\cdot}(t) + \alpha^{\cdot}(t)x \quad (|x| \leqslant a, t > 0, \tag{3.7}$$

$$Rlp(x, 0) + \frac{1}{\pi\theta} \int_{-a}^{a} p(\xi, 0) K\left(\frac{\xi - x}{H}\right) d\xi = \delta(0) + \alpha(0)x - f(x) \quad (|x| \leqslant a, t = 0).$$

The constant C in (3.7) has the form (3.3). To Eqs. (3.7) we must adjoin the equilibrium conditions (2.3).

Making Eqs. (3.7) and (2.3) dimensionless and introducing the notation of Sec. 2, in which we need only replace ρg with $(Rl)^{-1}$, we have

†In (3.4) we have taken account of the fact that u = 0 on the upper face of the layer.

$$D\,[\varphi(x,t)]^m+\varphi^{\cdot}(x,t)+\frac{\varkappa}{\pi}\int\limits_{-1}^{1}\varphi^{\cdot}(\xi,t)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi=\gamma^{\cdot}(t)+\alpha^{\cdot}(t)\,x\quad(t>0), \tag{3.8}$$

$$\varphi(x,0)=\frac{\varkappa}{\pi}\int\limits_{-1}^{1}\varphi(\xi,0)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi=\gamma(0)+\alpha(0)\,x-f(x)\quad(t=0),$$

$$|x|\leqslant 1,\ \varkappa=a(\theta Rl)^{-1},\ \Lambda=Ha^{-1},$$

$$N(t)=\int\limits_{-1}^{1}\varphi(x,t)\,dx,\ N(t)\,e=\int\limits_{-1}^{1}x\varphi(x,t)\,dx.$$

4. Before proceeding to solve the system of equations (3.8), we note that by virtue of the relations (3.5) and the results of [5], we can state the following:

1. The operator

$$M\varphi=\frac{\varkappa}{\pi}\int\limits_{-1}^{1}\varphi(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi \tag{4.1}$$

is a self-adjoint operator which is completely continuous and positive-definite and acts from $L_2(-1,\ 1)$ into $L_2(-1,\ 1)$.

2. If the function $f(x)\in L_2(-1,\ 1)$, then the solution of the second equation in (3.8) in the space $L_2(-1,\ 1)$ exists and is unique for any values of the parameters $\varkappa,\ \Lambda\in(0,\ \infty)$.

3. The eigenvalues η_n of operator (4.1) are real and positive, and $\eta_1\geqslant\eta_2\geqslant\ldots\geqslant\eta_n\geqslant\ldots,\ \eta_n\sim n^{-1}\ln n\ (n\to\infty)$.

We construct the asymptotic solution of the system of equations (3.8) for small values of time subject to the condition that for N(t) in a neighborhood of t = 0 we have the expansion (2.5). From this point on we may, without loss of generality, consider the case of a plane die (α = e = 0). We represent $\gamma(t)$ in the form (2.6), and we shall seek $\varphi(x,\ t)$ in the form

$$\varphi(x,\ t)=\varphi_0(x)+\varphi_1(x)\,t+\ldots+\varphi_n(x)t^n+O(t^{n+1}), \tag{4.2}$$

where $\varphi_0(x)=\varphi(x,\ 0)$ and is determined from the second equation in (3.8).

Substituting (2.5), (2.6), and (4.2) into Eqs. (3.8) and equating terms on the left and right sides that have equal powers of t, we obtain

$$\int\limits_{-1}^{1}\varphi_n(x)\,dx=N_n\ (n=0,1,2,\ldots), \tag{4.3}$$

$$\varphi_1(x)+\frac{\varkappa}{\pi}\int\limits_{-1}^{1}\varphi_1(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi=\gamma_1-D\varphi_0^m(x)\quad(|x|\leqslant 1),$$

$$\varphi_2(x)+\frac{\varkappa}{\pi}\int\limits_{-1}^{1}\varphi_2(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi=\gamma_2-\frac{1}{2}Dm\varphi_0^{m-1}(x)\,\varphi_1(x)\quad(|x|\leqslant 1),\ \ldots$$

The integral equations of the second kind in (4.3), by virtue of the properties of operator (4.1), are Fredholm equations [6] and, when taken together with the second equation in (3.8), serve to determine $\varphi_n(x)$ (n = 0, 1, 2, ...) on the assumption that the γ_n are given. The latter can then be determined from the integral relations (4.3). Approximate solutions of the integral equations can be found, for example, by the method of [5].

Now we construct the asymptotic solution of the system of equations (3.8) for large values of time, subject to the condition that for N(t) in a neighborhood of t = ∞ we have the expansion (2.8). As before, we consider as an example the case of a plane die (α = e = 0). We represent $\gamma(t)$ in the form (2.9), and we shall seek $\varphi(x,\ t)$ in the form†

$$\varphi(x,\ t)=\varphi_\infty[1+\varphi_1(x)e^{-Dt}+\ldots+\varphi_n(x)e^{-Dnt}+O(e^{-D(n+1)t})]. \tag{4.4}$$

†This structure of the solution is based on a report by V. M. Aleksandrov and E. V. Kovalenko: "Contact problems in the theory of elasticity in the case of nonlinear wear," in: Contact Rigidity in Instrument-Making and Machine Construction [in Russian], Riga Polytechnic Institute, Riga (1979).

Substituting (2.8), (2.9), and (4.4) into the system of equations (3.8) and equating in the relations so obtained the coefficients of equal powers of e^{-Dt} on the left and right sides, we find

$$\varphi_\infty = \frac{1}{2} N_0, \ \varphi_\infty \int_{-1}^{1} \varphi_n(x)\,dx = N_n \quad (n=1, 2, \ldots), \ D\varphi_\infty^m = \gamma_1,$$

$$\left(1-\varphi_\infty^{m-1}\right)\varphi_1(x) + \frac{\varkappa}{\pi}\int_{-1}^{1}\varphi_1(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = \beta_1\varphi_\infty^{-1} \quad (|x|\leqslant 1), \tag{4.5}$$

$$\left(1-\frac{1}{2}m\varphi_\infty^{m-1}\right)\varphi_2(x) + \frac{\varkappa}{\pi}\int_{-1}^{1}\varphi_2(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = \beta_2\varphi_\infty^{-1} + \frac{1}{4}m(m-1)\varphi_\infty^{m-1}\varphi_1^2(x) \quad (|x|\leqslant 1), \ldots$$

The Fredholm integral equations of the second kind in (4.5) serve to determine the $\varphi_n(x)$ ($n = 1, 2, \ldots$) on the assumption that the β_n are given. The latter can then be determined from the integral relations in (4.5). Approximate solutions of the integral equations (4.5) can be found, as noted earlier, by the method of [5].

Now we shall assume that there exists a value $t = T$ such that $\varphi_+(x, T) = \varphi_-(x, T)$, where $\varphi_\pm(x, t)$ are the contact stresses obtained for large and small values of time, respectively, by formulas (4.2) and (4.4). Then we also have $\gamma_+(T) = \gamma_-(T)$, from which we can determine the constant γ_0 in (2.9).

It should be noted that in a number of cases it is sometimes more useful to work with another algorithm for constructing the solution of the system (3.8) for large values of time. We shall describe this algorithm for the case N = const.

We represent $\gamma(t)$ in the form

$$\gamma(t) = \gamma_1 t + \gamma_0 + \sum_{k=1}^{\infty} e^{-\delta_k t} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \zeta_{kn}\, e^{-(\delta_k+\delta_n)t} + \ldots \tag{4.6}$$

The specification of $\gamma(t)$ in the form (4.6) is justified by physical considerations and is based on the above-mentioned report by the authors.

We shall seek a solution of the integral equation (3.8) in the form

$$\varphi(x, t) = \varphi_\infty\{1 + \varphi_1(x, t) + \ldots + \varphi_n(x, t) + o[\varphi_n(x, t)]\}. \tag{4.7}$$

Now, using an analog of Newton's method [6], we obtain

$$D\varphi_\infty^m = \gamma_1; \tag{4.8}$$

$$Dm\varphi_\infty^{m-1}\varphi_1(x,t) + \varphi_1^{\cdot}(x,t) + \frac{\varkappa}{\pi}\int_{-1}^{1}\varphi_1^{\cdot}(\xi,t)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = -\varphi_\infty^{-1}\sum_{k=1}^{\infty}\delta_k\, e^{-\delta_k t} \quad (|x|\leqslant 1), \tag{4.9}$$

$$Dm\varphi_\infty^{m-1}\left[\varphi_2(x,t) + \frac{m-1}{2}\varphi_1^2(x,t)\right] + \varphi_2^{\cdot}(x,t) +$$
$$+\frac{\varkappa}{\pi}\int_{-1}^{1}\varphi_2^{\cdot}(\xi,t)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = -\varphi_\infty^{-1}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\zeta_{kn}(\delta_k+\delta_n)\,e^{-(\delta_k+\delta_n)t} \ (|x|\leqslant 1), \ldots$$

We represent the solutions of the integral equations (4.9) in the form

$$\varphi_1(x,t) = \sum_{k=1}^{\infty} s_k(x)\, e^{-\delta_k t}, \ \varphi_2(x,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} s_{kn}(x)\, e^{-(\delta_k+\delta_n)t}, \ldots \tag{4.10}$$

Then, after some obvious transformations, we can write

$$\frac{\varkappa}{\pi}\int_{-1}^{1} s_n(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = \alpha_n s_n(x) + \varphi_\infty^{-1} \quad (|x|\leqslant 1), \tag{4.11}$$

$$\frac{\varkappa}{\pi}\int_{-1}^{1} s_{kn}(\xi)\,K\left(\frac{\xi-x}{\Lambda}\right)d\xi = \alpha_{kn}s_{kn}(x) + \frac{m-1}{2}(1+\alpha_{kn})\,s_k(x)\,s_n(x) + \varphi_\infty^{-1}\zeta_{kn} \ (|x|\leqslant 1), \ldots$$

$$\alpha_n = \frac{Dm\varphi_\infty^{m-1} - \delta_n}{\delta_n}, \ \alpha_{kn} = \frac{\alpha_k\alpha_n - 1}{2+\alpha_k\alpha_n}.$$

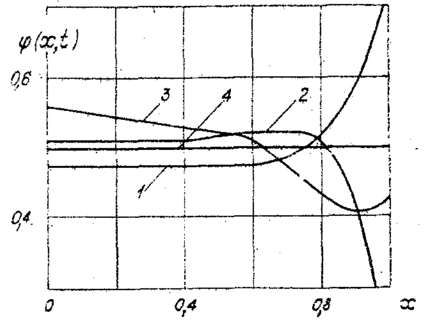

Fig. 3

It should be noted that by virtue of the above-mentioned properties of the operator M, Eqs. (4.11), for almost all $\alpha_k (k \geqslant 1)$ and $\Lambda \in (0, \infty)$, will be uniquely solvable in $L_2(-1, 1)$. Furthermore,

$$N = \int_{-1}^{1} \varphi(x, t)\, dx = N_0 + \sum_{n=1}^{\infty} N_n e^{-\delta_n t} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} N_{kn} e^{-(\delta_k + \delta_n)t} + \ldots, \tag{4.12}$$

$$N_0 = 2\varphi_\infty,\ N_n = \varphi_\infty \int_{-1}^{1} s_n(x)\, dx = 0,\ N_{kn} = \varphi_\infty \int_{-1}^{1} s_{kn}(x)\, dx, \ldots$$

We shall seek the solutions of Eqs. (4.11) in the form of Fourier series in an orthonormal system of Legendre polynomials:

$$s_k(x) = \sqrt{2}\, \varphi_\infty^{-1} \sum_{n=0}^{\infty} a_n^k P_{2n}^*(x),\ P_n^*(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x), \tag{4.13}$$

$$s_{kn}(x) = 2\varphi_\infty^{-2} \sum_{j=0}^{\infty} b_j^{kn} P_{2j}^*(x).$$

If we also take into account the equations

$$K\left(\frac{\xi - x}{\Lambda}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e_{ij}(\Lambda)\, P_{2i}^*(\xi)\, P_{2j}^*(x), \tag{4.14}$$

$$s_k(x)\, s_n(x) = \sum_{j=0}^{\infty} r_j^{kn} P_{2j}^*(x)$$

(the form of the coefficients $e_{ij}(\Lambda)$, r_j^{kn} is given in [5]), substitute (4.13) and (4.14) into (4.11), use the condition of orthogonality of the Legendre polynomials, and equate in the resulting relations the coefficients of the right and left sides for polynomials with the same number, we obtain

$$\frac{\varkappa}{\pi} \sum_{j=0}^{\infty} e_{ij}(\Lambda)\, a_j^k - \alpha_k a_i^k = \delta_{i0} \quad (i = 0, 1, \ \ldots); \tag{4.15}$$

$$\frac{\varkappa}{\pi} \sum_{j=0}^{\infty} e_{ij}(\Lambda)\, b_j^{kn} - \alpha_{kn} b_i^{kn} = \frac{m-1}{2}(1 + \alpha_{kn})\, r_i^{kn} + \varphi_\infty \frac{1}{\sqrt{2}} \zeta_{kn} \delta_{0i} \tag{4.16}$$

$$(i = 0, 1, 2, \ldots),$$

where δ_{ij} is the Kronecker delta.

It should be noted that, by (4.12) and (4.13), $a_0^k = 0$, $b_0^{kn} = 0$ $(k, n \geqslant 1)$. These conditions serve to determine the unknown quantities δ_n, ζ_{kn}. To see this, we observe that from (4.15) we have $a_0^k = \Delta_1 \Delta^{-1}$, where Δ is the fundamental determinant of the system (4.15) and Δ_1 is an auxiliary determinant obtained from Δ by replacing its first column with the elements $\{1, 0, 0, \ldots, 0, \ldots\}$. The determinant Δ_1 is symmetric, and therefore its roots α_j $(j = 1, 2, \ldots)$ are real. Furthermore, taking into account the results of [7], we can assert that $\eta_{2j+2} < \alpha_j < \eta_{2j}$, where η_j are the eigenvalues of the integral operator (4.1). This provides

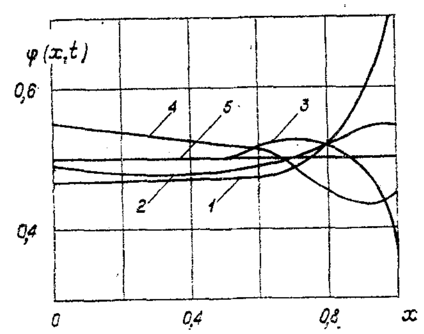

Fig. 4

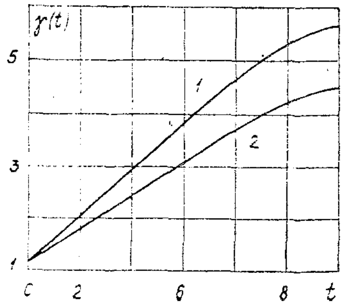

Fig. 5

the justification for the structure of the solution (4.6), (4.7) of the integral equation (3.8).

Now we shall show that the condition $b_0^{kn} = 0$ serves to determine uniquely the value of ζ_{kn} in (4.6). From the system (4.16) it follows that $b_0^{kn} = \Delta_3\Delta_2^{-1}$, where $\Delta_2 = \det[\varkappa\pi^{-1}e_{ij}(\Lambda) - \alpha_{kn}\delta_{ij}]$ and Δ_3 is an auxiliary determinant of the system (4.16) obtained from the fundamental determinant Δ_2 by replacing the first column with the elements $(m-1)2^{-1}(1+\alpha_{kn})\{r_0^{kn} + \sqrt{2}\zeta_{kn}(m-1)^{-1}(1+\alpha_{kn})^{-1}, r_1^{kn}, r_2^{kn}, \ldots, r_i^{kn}, \ldots\}$. Obviously in order to select ζ_{kn} uniquely from the condition $b_0^{kn} = 0$ we must have a determinant $\Delta_4 = \det[\varkappa\pi^{-1}e_{ij}(\Lambda) - \alpha_{kn}\delta_{ij}]$ $(i, j \geqslant 1)$ which is different from zero. It is not difficult to see that Δ_4 has the same structure as Δ_1, in which instead of the elements $\varkappa\pi^{-1}e_{ij}(\Lambda) - \alpha_k\delta_{ij}$ we have $\varkappa\pi^{-1}e_{ij}(\Lambda) - \alpha_{kn}\delta_{ij}$. Assume that $\Delta_4 = 0$: then from the condition $a_0^k = 0$ it follows that $\alpha_{kn} = \alpha_n$, and hence $\alpha_n^2 = -3$ $(n \geqslant 1)$. But this is impossible, since $\alpha_n > 0$ $(n \geqslant 1)$. Consequently $\Delta_4 \neq 0$, and the condition $b_0^{kn} = 0$ serves to guarantee a unique choice of ζ_{kn} in (4.6).

Having determined the numbers α_n, ζ_{kn} in this manner, we next find the values of a_j^k, b_j^{kn} from the inhomogeneous systems (4.15), (4.16) and construct the functions $s_n(x)$, $s_{kn}(x)$. It should be noted that these systems, by virtue of the properties of the operator M and Hilbert's theorem [6], are uniquely solvable in the space of square summable sequences l_2 for any values of the parameter $\Lambda \in (0, \infty)$, and the method of reduction can be used for solving them.

The constant γ_0 in (4.6) is determined, as above, from the condition $\gamma_+(T) = \gamma_-(T)$.

5. We give a numerical illustration of the proposed algorithms by using the example of the second problem in Sec. 3. Suppose that the lower elastic layer is rigidly fixed to the base [4]

$$L(u) = \frac{2\eta \operatorname{sh} 2u - 4u}{u(2\eta \operatorname{ch} 2u + 1 + \eta^2 + 4u^2)}, \quad \eta = 3 - 4\sigma, \tag{5.1}$$

and the force impressing the die is constant with respect to time, N = 1. Then in the expansions (4.3) and (4.5), respectively, we have

$$N_0 = 1, \ \varphi_\infty = 1/2, \ N_n = 0$$
$$(n = 1, 2, \ldots).$$

In (3.8) and (5.1) we set $D = 1$, $\varkappa = \pi$, $\Lambda = 1$, $\sigma = 0.3$.

In the asymptotic expansions for $\varphi(x, t)$, (4.2) and (4.7), we retain only the first two terms. The law of distribution of the contact normal stresses $\varphi(x, t)$ as a function of x and t is shown in Fig. 3 (m = 1.5, with curves 1-4 corresponding to t = 0, 3, 9, and ∞) and Fig. 4 (m = 2, with curves 1-5 corresponding to t = 0, 1, 2, 10, and ∞). It should be noted that the asymptotic expansion (4.2) for $\varphi(x, t)$ constructed for sufficiently small values of time coincides with the corresponding asymptotic expansion (4.7) constructed for sufficiently large values of t when $T \approx 8$ in the case m = 1.5 and when $T \approx 8.5$ in the case m = 2. Furthermore, a numerical analysis of the problem showed that for m = 1.5 and m = 2 and equal values of x and t, the values of $\varphi(x, t)$ differ from each other by no more than 5-6%.

To determine the constant γ_0 in formula (4.6) we use, as noted above, the condition $\gamma_+(T) = \gamma_-(T)$. We shall have for m = 1.5 and m = 2, respectively, the values $\gamma_0 = 1.483$ and

$\gamma_0 = 1.577$. Figure 5 shows the law of distribution of $\gamma(t)$ as a function of t when m = 1.5 and 2 (curves 1 and 2, respectively). It should be noted that this function is almost linear. In addition, the set under the die will be larger for lower nonlinearity indexes m if other conditions are equal.

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NUMERICAL ASYMPTOTIC SOLUTION OF STRENGTH AND VIBRATIONS PROBLEMS OF THIN SHELLS OF REVOLUTION

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UDC 539.3

For thin shells of revolution whose middle surface has a nonnegative Gaussian curvature, a numerical analytical approximate solution is constructed for the class of linear boundary-value problems allowing of separation of variables.

It is known that the solution of each such problem decomposes into a slowly varying part and a solution of edge effect type. On this basis, a method of construction the approximate solution of the problem is proposed in [1, 2], where it is proposed to seek the slowly varying part of the solution by a numerical method, and the edge effects by an asymptotic method. On the basis of this method, an algorithm is constructed in this paper, which can be applied to a broader class of problems as compared to [1] because of utilization of the method of elimination in the boundary conditions [3]. As an illustration of the method, solutions are presented for a number of strength and vibrations problems for shells of different geometries.

1. Many strength and vibration problems for elastic shells of revolution reduce to seeking solutions of a particular kind

$$u_\sigma^{mn} = \exp(i\omega_m t + inx_2)\, U_\sigma^{mn}(x_1), \quad w^{mn} = \exp(i\omega_m t + inx_2) W^{mn}(x_1). \tag{1.1}$$

Here t is the time; x_1, x_2, orthogonal coordinates of the shell middle surface; $i = \sqrt{-1}$; ω_m, real integers; and m and n, integers, the subscript σ takes on the values 1 and 2; u_1, u_2, w, displacements in the x_1, x_2 directions and along the external normal. The well-developed apparatus of shallow shell theory [4] can be applied to describe solutions of the form (1.1) with $n \geqslant 4$. In the absence of tangential components of the surface forces, a force function $\Psi(x_1, x_2, t)$ is introduced. By virtue of (1.1) we will have

$$\varphi^{mn}(x_1, x_2, t) = \exp(i\omega_m t + inx_2)\Phi^{mn}(x_1). \tag{1.2}$$

The system of governing equations of shallow shell theory after making the system operators dimensionless and substituting functions from (1.1) and (1.2) becomes

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